Strong gravity Lense-Thirring Precession in Kerr and Kerr-Taub-NUT spacetimes

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An exact expression derived in the literature for the rate of dragging of inertial frames (Lense-Thirring (LT) precession) in a general stationary spacetime, is reviewed. This formula is valid beyond the weak field approximation. When used for the Kerr metric, it leads to the LT precession frequency in the *strong gravity* regime appropriate to compact gravitating objects like rotating neutron stars and black holes. Numerical values of the precession rate are computed for a few known cases of pulsars and compared to the precession rates in the weaker gravity regimes of the earth and the sun. We also derive the exact LT precession rates in Kerr-Taub-NUT and Taub-NUT spacetime to show that in the case of spherically symmetric, zero angular momentum NUT spacetimes, the frame-dragging effect *does not* vanish.

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I. INTRODUCTION

Stationary spacetimes with angular momentum (rotation) are known to exhibit an effect called Lense-Thirring (LT) precession whereby locally inertial frames are dragged along the rotating spacetime, making any test gyroscope in such spacetimes *precess* with a certain frequency called the LT precession frequency [1]. This frequency has been shown to decay as the inverse cube of the distance of the test gyroscope from the source [2] for large enough distances where curvature effects are small, and known to be proportional to the angular momentum of the source. The largest precession frequencies (Ω) are thus expected to be seen very close to the source (like the surface of a neutron star or a pulsar, or the horizon of a black hole), as well as for spacetimes rotating very fast with large angular momenta.

Earlier analyses of the LT effect [3] assume large distances ($r > M$, M is the ADM mass of the rotating spacetime due to a compact object like a neutron star) for the test gyroscope. Further, the rotating spacetime solution is usually approximated as a Schwarzschild spacetime, and the effect of rotation is confined to a perturbative term added on to the Schwarzschild metric. Basically, this is tantamount to reducing the spacetime metric to a flat metric (Minkowski spacetime) together with the added perturbation. This leads to the standard result for LT precession frequency in the weak field approximation, given by [3, 4]

$$\vec{\Omega}_{LT} = \frac{1}{r^3}[3(\vec{J}.\hat{r})\hat{r} - \vec{J}] \tag{1}$$

where, $\hat{r}$ is the unit vector along r direction. In a recent work reported in ref. [5], an alternative approach based on solving the geodesic equations of the test gyroscope numerically, *once again* within the weak gravitational field approximation, is used to compute the frame-dragging effect for galactic-centre black holes.

In another very recent related work [6], Hackman and Lämmerzahl have given an expression of LT precession valid up to *first order* in the Kerr parameter a for a general axially symmetric Plebanski-Demianski spacetime. This is obviously a good approximation for slowly-rotating compact objects. The LT precession rate has also been derived [7, 8] through solving the geodesic equations for both Kerr and Kerr-de-Sitter spacetimes at the *polar orbit* in terms of Appell's first hypergeometric function. These results are not applicable for orbits which lie in orbital planes other than the polar plane. We understand that observations of precession [9] due to locally inertial frame-dragging have so far focused on spacetimes where the curvatures are small enough; e.g., the LT precession in the earth's gravitational field which was probed recently by Gravity Probe B. There has been so far no attempt to measure LT precession effects due to frame-dragging in *strong gravity regimes* [10].

While most textbook calculations of the LT precession rate focus usually on the weak-field approximation, an exception is the classic work of Misner, Thorne and Wheeler, [11]. Here, the orbital angular velocity for *locally non-rotating* observers in a Kerr-Newman spacetime is given in eqn.(33.24) as an exercise. This formula does not appear to be restricted to the weak-field approximation. However, from an astrophysical standpoint, it is not clear that the computed angular frequency corresponds to what might be measured as the Lense-Thirring precession in a strong gravity situation. Because it has been derived in a locally non-rotating frame which the authors amply clarify is *not* a *'Copernican' frame*, (A Copernican frame is a local orthonormal tetrad at rest (so moving only in the "time" direction determined by the timelike Killing vector of the spacetime) and "locked" to the spatial part of whatever such coordinate system is chosen, so that it is "at rest" with respect to the local inertial frames at infinity. Within this frame, an untorqued gyro in a stationary but not static spacetime held fixed by a support force applied to its center of mass precesses. Since the Copernican frame does not rotate (by construction) relative to the inertial frames at asymptotic infinity ("fixed stars"), the observed precession rate in the Copernican frame also gives the precession rate of the gyro relative to the fixed stars.) it is not obvious that this angular frequency corresponds to the LT precession rate of a test gyroscope in a stationary (but not static) spacetime. A naïve limiting procedure does not appear to reduce this frequency to the standard weak-field result (1) in 'Copernican' frames quoted in most other textbooks for the LT precession rate in a weak gravitational field. For this reason, a rederivation of the LT precession rate in spacetime regions of large curvature, from a *'Copernican' frame* perspective, remains, in our opinion, an important issue. Two motivating factors may be cited in support of such a contention.

First of all, the near-horizon physics of black holes and that of the outer layers of neutron stars emitting X-rays from their accretion discs also might need to be reanalyzed in view of the nontrivial LT precession of test geodesics in their vicinity. Most extant treatments of accretion disc physics appear to consider these effects as unimportant, even though there is not much in the way of a computation to theoretically justify this viewpoint. With upcoming X-ray observatories, as well as multi-wavelength strong gravity space probes currently under construction, which envisage to make observations of possible frame-dragging effects in strong gravity situations in the near future, the need to go beyond the weak field approximation is paramount. A recent work by Stone and Loeb [12] has estimated the effect of weak-field LT precession on accreting matter close to compact accreting objects. While there are claims that what has been estimated in this work pertains more to orbital precession, rather than precession of a test gyroscope (which remains the classic example of LT precession), it is obvious that in the vicinity of the spacetime near the surface of neutron stars (respectively, the horizons of black

holes), the large LT precession of test gyroscopes ought to manifest in changes in the predicted X-ray emission behaviour originating from modifications in the behaviour of infalling timelike geodesics of accreting matter particles due to LT precession. Thus, there is sufficient theoretical motivation to compute LT precession rates in the strong gravity regime, in a bid towards a prediction that future probes of the inertial frame dragging effect, in such a regime, may correlate with.

For rapidly-rotating radio-pulsars, even though the so-called 'lighthouse mechanism' provides a first order mechanism of pulsating radio emissions, the theoretical framework employed often tends to ignore most general relativistic complications, most importantly that of inertial frame dragging near the surface of these neutron stars. Without an understanding of LT precession in strong gravity situations, these effects cannot be accurately estimated and blithely ignoring them might turn out to be erroneous.

In this paper, we derive the LT precession rate in a strong gravity regime in a 'Copernican' frame. Our result is applicable to all stationary spacetimes, no matter whether they are axisymmetric or not. This result is also applicable for all orbits (not only for polar and equatorial orbits), located at various distances and different angles. In this sense, it is rather more general than the result of [11]. Also, the oft-quoted weak-field result (1) (in a 'Copernican' frame) for the LT precession rate is readily obtained from this general result, inserting the metric for the desired spacetime.

The paper is organized as follows : in section II, we discuss about some special features of the frame-dragging effect in a *stationary* spacetime within a 'Copernican' frame. In section III, the *exact* precession rate of the LT precession in Kerr spacetime [13], without invoking either the weak gravity approximation or an approximation involving the Kerr parameter. The weak-field approximation reported everywhere in extant literature is then shown to emerge from our general formulation, in section IV. We then compute in section V, numerical values of the precession frequency in the strong gravity regime and compare these to the standard weaker gravity numerical values, for various gravitating bodies, like the earth, the sun and several known pulsars, including the millisecond Hulse-Taylor binary pulsar. In section VI, we derive the exact LT precession rates in Kerr-Taub-NUT and Taub-NUT spacetimes. We conclude in section VII with a summary and an outlook for future work.

II. EXACT LENSE-THIRRING PRECESSION FREQUENCY

The exact LT frequency of precession of test gyroscopes in strongly curved stationary spacetimes, analyzed within a 'Copernican' frame, is expressed as a co-vector given in terms of the timelike Killing vector fields K of the stationary spacetime, as (in the notation of ref. [14])

$$\tilde{\Omega} = \frac{1}{2K^2} * (\tilde{K} \wedge d\tilde{K})$$

or,

$$\Omega_\mu = \frac{1}{2K^2} \eta_\mu{}^{\nu\rho\sigma} K_\nu \partial_\rho K_\sigma \ , \tag{2}$$

where, $\eta^{\mu\nu\rho\sigma}$ represent the components of the volume-form in spacetime and $\tilde{K}$ & $\tilde{\Omega}$ denote the one-form dual to K & Ω, respectively. Note that $\tilde{\Omega}$ vanishes if and only if $(\tilde{K} \wedge d\tilde{K}) = 0$. This happens only for a static spacetime.

Using the coordinate basis form of $K = \partial_0$, the co-vector components are easily seen to be $K_\mu = g_{\mu 0}$. The combination in the right-hand-side of eqn.(2) may now be expressed in terms of the metric components

$$\tilde{\Omega} = \frac{1}{2} \frac{\epsilon_{ijl}}{g_{00}\sqrt{-g}} [g_{0i,j} (g_{00}g_{kl} - g_{0k}g_{0l}) - g_{0i}g_{kl}g_{00,j}]dx^k \tag{3}$$

The vector field corresponding to the LT precession co-vector in (3) can be expressed in coordinate basis as

$$\Omega = \frac{1}{2} \frac{\epsilon_{ijl}}{\sqrt{-g}} \left[g_{0i,j} \left(\partial_l - \frac{g_{0l}}{g_{00}} \partial_0 \right) - \frac{g_{0i}}{g_{00}} g_{00,j} \partial_l \right] \tag{4}$$

The remarkable feature of the above equation (4) is that it is applicable to any arbitrary stationary spacetime (irrespective of whether it is axisymmetric or not); it gives us the exact rate of LT precession in such a spacetime. For instance, a Newman-Unti-Tamburino (NUT) spacetime with vanishing ADM mass is known to be spherically symmetric, but still has an angular momentum (dual or 'magnetic' mass [15],[16]); we use eqn.(4) to compute the LT precession frequency in this case as well. This result is new and rather general, because, there is only one constraint on the spacetime : that it must be stationary, which is the only necessary condition for LT precession. The utility of this equation is that; if any metric $(g_{\mu\nu})$ contains all 10 (4×4) elements non-vanishing, it can be used to calculate the LT precession in that spacetime. In this case, the precession rate depends only on non-zero $g_{0i}(i = 1, 2, 3)$ components, not on any other non-zero off-diagonal components of

the metric. So, this equation also reveals that the LT precession rate is completely determined by the metric components $g_{0\mu}(\mu = 0, i)$, and is quite independent of the other components (in co-ordinate basis).

A careful inspection of (4) reveals that the LT precession rate given by it vanishes under the following conditions :

- If g_{0i} vanishes, because this corresponds to a *static* spacetime.

- If the line element of a stationary spacetime is of the form,

$$ds^2 = g_{00}(x^m)dx^0dx^0 + 2g_{0m}(x^m)dx^0dx^m + g_{ij}dx^idx^j \ . \tag{5}$$

Here, $m = 1$ or 2 or 3 and $x^m = x^1$ or x^2 or x^3. We can explore it with an example. Suppose, $x^m = r$ and then line element would be

$$ds^2 = g_{00}(r)dx^0dx^0 + 2g_{0r}(r)dx^0dr + g_{ij}dx^idx^j \tag{6}$$

where, $g_{00}(r), g_{0r}(r)$ mean that g_{00}, g_{0r} are the functions of only r. In such a situation, the formula (eqn.4) would be

$$\Omega = \frac{1}{2}\frac{\epsilon_{rjl}}{\sqrt{-g}}\left[g_{0r,j}\left(\partial_l - \frac{g_{0l}}{g_{00}}\partial_0\right) - \frac{g_{0r}}{g_{00}}g_{00,j}\partial_l\right] \tag{7}$$

But, we see that g_{00}, g_{0r} are the functions of r only. So,

$$\epsilon_{rjl}g_{0r,j} = \epsilon_{rjl}\frac{\partial g_{0r}(r)}{\partial x^j} = 0 \tag{8}$$

for $j \neq r$ and for $j = r$, ϵ_{rrl} vanish due to the antisymmetry properties of the Levi-Civita tensor density. The 2nd term $\epsilon_{rjl}g_{0r}g_{00,j} = \epsilon_{rrk}g_{0r}g_{00,r} = 0$ for the same reason mentioned in the above. So, this is a very interesting situation in where LT precession rate vanishes, even though $g_{0i} \neq 0$. In the above given example if we replace $x^m = \phi$ instead of $x^m = r$, the LT precession vanishes. A well-known example is the Kerr metric in Boyer-Lindquist coordinates: $g_{0\phi}$ is not a function of ϕ. It is the function of $(g_{0\phi} = g_{0\phi}(r, \theta))$ only r and θ. Above all, we know that Kerr metric is axisymmetric. So, it is impossible to bring the Kerr metric into the form of the line element (5), and the LT precession in such a spacetime cannot vanish.

The general formula (eqn.4) of LT precession rate shows that any stationary non-static spacetime (axisymmetric or not) have a non-zero precession rate. One may ask why the stationary spacetimes show non-zero LT precession rate even if it is non-axisymmetric.

A spacetime is said to be stationary if it possesses a timelike Killing vector field ξ^a; further, a stationary spacetime is said to be static if there exists a spacelike hypersurface Σ which is orthogonal to the orbits of the timelike isometry. By Frobenius's theorem of hypersurface orthogonality, we can write for a static spacetime,

$$\xi_{[a}\nabla_b\xi_{c]} = 0 \tag{9}$$

If $\xi^a \neq 0$ everywhere on Σ, then in a neighbourhood of Σ, every point will lie on a unique orbit of ξ^a which passes through Σ. From the explicit form of a static metric, it can be seen that the diffeomorphism defined by $t \to -t$ (the map which takes each point on each Σ_t to the point with the same spatial coordinates on Σ_{-t}), is an isometry. The "time translation" symmetry, $t \to t + constant$ is possessed by all stationary spacetimes. Static spacetimes on the other hand, possess an additional symmetry, "time reflection" symmetry over and above the "time translation symmetry". Physically, the fields which are time translationally invariant can fail to be time reflection invariant if any type of "rotational motion" is involved, since the time reflection will change the direction of rotation. For example, a rotating fluid ball may have a time-independent matter and velocity distribution, but is unable to possess a time reflection symmetry [17]. In the case of stationary spacetimes, the failure of the hypersurface orthogonality condition (eqn.9) implies that neighbouring orbits of ξ^a "twist" around each other. The twisting of the orbits of ξ^a is the cause of that *extra precession* in stationary non-static spacetimes.

III. EXACT LENSE-THIRRING PRECESSION RATE IN KERR SPACETIME

One can now use eqn. (4) to calculate the angular momentum of a test gyroscope in a Kerr spacetime to get the LT precession in a strong gravitational field. In Boyer-Lindquist coordinates, the Kerr metric is written as,

$$ds^2 = -\left(1 - \frac{2Mr}{\rho^2}\right)dt^2 - \frac{4Mar\sin^2\theta}{\rho^2}d\phi dt + \frac{\rho^2}{\Delta}dr^2 + \rho^2 d\theta^2 + \left(r^2 + a^2 + \frac{2Mra^2\sin^2\theta}{\rho^2}\right)\sin^2\theta d\phi^2 \tag{10}$$

where, a is Kerr parameter, defined as $a = \frac{J}{M}$, the angular momentum per unit mass and

$$\rho^2 = r^2 + a^2 \cos^2\theta, \Delta = r^2 - 2Mr + a^2 \tag{11}$$

For the Kerr spacetime, the only nonvanishing $g_{0i} = g_{0\phi}$, $i = \phi$ and $j, l = r, \theta$; substituting these in eq. (4), the precession frequency vector is given by

$$\Omega_{LT} = \frac{1}{2\sqrt{-g}} \left[\left(g_{0\phi,r} - \frac{g_{0\phi}}{g_{00}} g_{00,r} \right) \partial_\theta - \left(g_{0\phi,\theta} - \frac{g_{0\phi}}{g_{00}} g_{00,\theta} \right) \partial_r \right] \tag{12}$$

where, the various metric components can be read off from eqn. (10). Likewise,

$$\sqrt{-g} = \rho^2 \sin\theta \tag{13}$$

In order to make numerical predictions for the LT precession frequency in a strong gravity domain, we need to transform the precession frequency formula from the coordinate basis to the orthonormal 'Copernican' basis: first note that

$$\Omega_{LT} = \Omega^\theta \partial_\theta + \Omega^r \partial_r \tag{14}$$
$$\Omega^2_{LT} = g_{rr}(\Omega^r)^2 + g_{\theta\theta}(\Omega^\theta)^2 \tag{15}$$

Next, in the orthonormal 'Copernican' basis at rest in the rotating spacetime, the tetrad vector $e_0 = u$, the tangent vector along the integral curve of the timelike Killing vector K. In this basis, with our choice of polar coordinates, Ω_{LT} can be written as

$$\vec{\Omega}_{LT} = \sqrt{g_{rr}} \Omega^r \hat{r} + \sqrt{g_{\theta\theta}} \Omega^\theta \hat{\theta} \tag{16}$$

where, $\hat{r}$ is the unit vector along the direction θ. For the Kerr metric,

$$\Omega^\theta = -J \sin\theta \frac{(\rho^2 - 2r^2)}{\rho^4(\rho^2 - 2Mr)} \tag{17}$$

$$\Omega^r = 2J \cos\theta \frac{r\Delta}{\rho^4(\rho^2 - 2Mr)} \tag{18}$$

Substituting the values of Ω^r and Ω^θ in eqn.(16), we get the following expression of LT precession rate in Kerr spacetime

$$\vec{\Omega}_{LT} = 2aM \cos\theta \frac{r\sqrt{\Delta}}{\rho^3(\rho^2 - 2Mr)} \hat{r} - aM \sin\theta \frac{\rho^2 - 2r^2}{\rho^3(\rho^2 - 2Mr)} \hat{\theta} \tag{19}$$

The magnitude of this vector is

$$\Omega_{LT}(r, \theta) = \frac{aM}{\rho^3(\rho^2 - 2Mr)} \left[4\Delta r^2 \cos^2\theta + (\rho^2 - 2r^2)^2 \sin^2\theta \right]^{\frac{1}{2}} \tag{20}$$

This is the LT precession rate where no weak gravity approximation has been made. It should therefore be applicable to any rotating spacetime like pulsars, rotating black hole etc.

IV.   WEAK FIELD LIMIT

For large distances, the Kerr metric is approximated as a Minkowski metric with the cross term $(g_{\phi t} d\phi dt)$, that is

$$ds^2 = ds^2_{Mink} - \frac{4Ma\sin^2\theta}{r} d\phi dt \tag{21}$$

and it has been shown that [3]

$$\vec{\Omega}_{LT} = \frac{1}{r^3} [3(\vec{J}.\hat{r})\hat{r} - \vec{J}] \tag{22}$$

For large distances and in weak gravitational fields (where we can take $r > M$ and $r > a$), the 2nd term of eqn.(19) reduces to

$$\sqrt{g_{\theta\theta}}\Omega^\theta \hat{r} = -\frac{J\sin\theta}{r^3(1 + \frac{a^2}{r^2}\cos^2\theta)^{\frac{3}{2}}} \frac{(-1 + \frac{a^2}{r^2}\cos^2\theta)\hat{r}}{(1 + \frac{a^2}{r^2}\cos^2\theta - \frac{2M}{r})}$$

$$\simeq \frac{J\sin\theta}{r^3}\hat{r} \tag{23}$$

Similarly, from the 1st term of eqn.(19) one obtains the following,

$$\sqrt{g_{rr}}\Omega^r \hat{r} = \frac{2J\cos\theta . r^2\sqrt{1 + \frac{a^2}{r^2} - \frac{2M}{r}}}{r^5(1 + \frac{a^2}{r^2}\cos^2\theta)^{\frac{3}{2}}\left(1 + \frac{a^2}{r^2}\cos^2\theta - \frac{2M}{r}\right)}\hat{}$$

$$\simeq \frac{2J\cos\theta}{r^3}\hat{\theta} \tag{24}$$

It follows that

$$\vec{\Omega}_{LT}(r,\theta) = \frac{J}{r^3}\left[2\cos\theta\hat{r} + \sin\theta\hat{\theta}\right] \tag{25}$$

where, θ is the colatitude. The resemblance of this equation with eq. (1) is unmistakeable.

The LT precession for a general stationary metric in the weak field limit may also be derived from (4). In this approximation,

$$g_{00} \simeq -1, g_{ij} = \delta_{ij}, \frac{g_{0l}}{g_{00}} << 1,$$

Under these conditions eqn.(4) reduces to,

$$\Omega \simeq \frac{1}{2}\epsilon_{ijl}g_{0i,j}\partial_l \tag{26}$$

As $e_l \simeq \partial_l$, a test gyroscope precesses in such a weak gravitational field with an angular velocity

$$\vec{\Omega} \simeq \frac{1}{2}\vec{\nabla} \times \vec{g}, \tag{27}$$

where, $\vec{g} \equiv (g_{01}, g_{02}, g_{03})$. Here, 1,2,3 indicate the space components in that spacetime.

V. PRELIMINARY COMPARISON WITH OBSERVATIONAL DATA

Let us take, as an example for the LT precession in strong gravitational field, the case of the Hulse-Taylor binary pulsar (also called PSR B1913+16) in the constellation of Aquila, located roughly 21,000 light-years far from the earth. Its mass, radius and rotational time period (T) are, respectively,

$$M = 1.44M_\odot = 2120.95 \text{ m},$$
$$R \approx 10 \text{ km(assuming)},$$
$$T = 59.03 \times 10^{-3} \text{ s} \tag{28}$$

respectively. Here, $M_\odot$ indicates the mass of the sun. Using these numerical values, we can easily calculate the angular velocity (ω) of the pulsar

$$\omega = \frac{2\pi}{59.03 \times 10^{-3}} = 105.96 \text{ rad s}^{-1}$$
$$= 3.53 \times 10^{-7} \text{ m}^{-1} \tag{29}$$

and Kerr parameter (taking moment of inertia $I \approx 0.4MR^2$, [18])

$$a \approx 14.13 \text{ m} \tag{30}$$

All numerical values are in gravitational units with $G = c = 1$.

Now, let us imagine a test gyroscope very close to the surface of the pulsar, in order to that we may capture the strong gravitational effects for LT precession. So, we take $r \to R$.

Massive objects	$\Omega_{strongLT}(\text{rad s}^{-1})$	$\Omega_{weakLT}(\text{rad s}^{-1})$
Earth	4.02×10^{-14}	4.02×10^{-14}
Sun	4.56×10^{-12}	4.56×10^{-12}
PSR B1257+12	239	178
PSR B1913+16	24	18
PSR J1614-2230	712	463
PSR J1748-2021B	274	121

TABLE I. Comparison between strong gravity and weak gravity LT precession (all these values are calculated at the pole ($\theta = 0$) and near the surface ($r \to R$) of that massive object).

Substituting the above values in eqn. (20) for $\theta = 0$ (which indicates that we are measuring the LT precession at the pole of the pulsar), we get

$$\Omega_{strongLT} = \frac{2JRc}{(R^2 + a^2)^{\frac{3}{2}}\sqrt{R^2 - 2MR + a^2}}$$
$$\approx 24 \text{ rad s}^{-1} \tag{31}$$

or,

$$\frac{\Omega_{strongLT}}{\omega} \approx 0.22 \tag{32}$$

On the other hand, if we use our well-known weak field approximation (eqn. no 25) for LT precession rate [19, 20], we get

$$\Omega_{weakLT} = \frac{2Jc}{R^3} \approx 18 \text{ rad s}^{-1} \tag{33}$$

or,

$$\frac{\Omega_{weakLT}}{\omega} \approx 0.17 \tag{34}$$

So, we can easily see that there is really a substantial ($\sim$ 30 %) difference between the *weak* and *strong* gravity LT precession rates in this case. But, if this precession rate for a gyroscope or a satellite due to the rotation of the earth or the sun, the numerical results from the *weak* and *strong* gravity LT precession rate formulas appear to agree to better than 1 %. So, for the earth or the sun, it does not matter whether one takes the strong gravity eqn.(19) or the weak gravity LT eqn.(25).

In Table 1, we offer a comparative numerical study between calculated values $\Omega_{strongLT}$ and Ω_{weakLT} for a few recently discovered pulsars, with the sun and the earth for comparison. The effect of strong gravity is as high as above 100 % in one particular example, and certainly above 30 % in general for all pulsars.

During the $\Omega_{strongLT}$ calculations, the backreaction effects of the binary companion of a particular pulsar can be usually neglected, because, the distance of this companion is in the order of 1 lt-sec(10^5 km) or more than that and Ω_{LT} is inversely proportional to ρ^3. The strong gravity LT effect computed which we focus on is significant near the surface of pulsars and near black hole event horizons. If the binary system under consideration has a non-compact companion of the pulsar which is reasonably distant, backreaction effects of this companion on LT precession of test gyroscopes may be safely ignored.

We should be careful in pointing out, however, that the table of comparisons above is merely to convey the importance of strong gravity effects in the rate of dragging of inertial frames in the case of compact gravitating objects; the actual numbers may have errors, both theoretical and observational, which have not been fully analysed yet.

VI.   LENSE-THIRRING PRECESSION IN KERR-TAUB-NUT SPACETIME

The Kerr-Taub-NUT spacetime is geometrically a stationary, axisymmetric vacuum solution of Einstein equation with Kerr parameter (a) and NUT charge (n). If the NUT charge vanishes, the solution reduces to the Kerr geometry. The metric of the Kerr-Taub-NUT spacetime is

$$ds^2 = -\frac{\Delta}{p^2}(dt - Ad\phi)^2 + \frac{p^2}{\Delta}dr^2 + p^2 d\theta^2 + \frac{1}{p^2}\sin^2\theta(adt - Bd\phi)^2 \tag{35}$$

With

$$\Delta = r^2 - 2Mr + a^2 - n^2, p^2 = r^2 + (n - a\cos\theta)^2,$$
$$A = a\sin^2\theta + 2n\cos\theta, B = r^2 + a^2 + n^2. \tag{36}$$

As the spacetime has an intrinsic angular momentum (due to Kerr parameter a), we can expect a non-zero frame-dragging effect. We get from eqn. (12), the LT precession rate in Kerr-Taub-NUT spacetime is

$$\vec{\Omega}_{LT}^{KTN} = \frac{\sqrt{\Delta}}{p}\left[\frac{a\cos\theta}{\rho^2 - 2Mr - n^2} - \frac{a\cos\theta - n}{p^2}\right]\hat{r} + \frac{a\sin\theta}{p}\left[\frac{r - M}{\rho^2 - 2Mr - n^2} - \frac{r}{p^2}\right]\hat{\theta} \tag{37}$$

In contrast to the Kerr spacetime, where the source of the LT precession is the Kerr parameter a, the Kerr-Taub-NUT spacetime has an extra somewhat surprising feature : the LT precession does not vanish even for vanishing Kerr parameter $a = 0$, so long as the NUT charge $n \neq 0$. This means that though the orbital angular momentum (J) of this spacetime vanishes, the spacetime does indeed exhibit an intrinsic *spinlike* angular momentum (at the classical level itself) which we discuss below in more detail. One can show that inertial frames are dragged along this orbitally non-rotating spacetime with the precession rate

$$\vec{\Omega}_{LT}^{TN} = \frac{n\sqrt{\Delta|_{a=0}}}{p^3}\hat{r} \tag{38}$$

where, $p^2 = r^2 + n^2$. Notice that the precession rate is independent of θ and also that it vanishes when the NUT charge vanishes, as already alluded to above. In fact, for $a = n = 0$, the Kerr-Taub-NUT spacetime reduces to the *static* Schwarzschild spacetime which of course does not cause any inertial frame dragging. We consider this curious phenomenon in somewhat more detail in the next subsection.

A. Lense-Thirring precession in NUT spacetime

The Taub-NUT spacetime is geometrically a stationary, spherically symmetric vacuum solution of Einstein equation with NUT charge (n). The Einstein-Hilbert action requires no modification to accommodate this NUT charge or "dual mass" which is perhaps an intrinsic feature of general relativity, being a gravitational analogue of a magnetic monopole in electrodynamics [21].

Consider the line element (of NUT spacetime), which is presented by Newman et. al.[15, 22, 23]

$$ds^2 = -f(r)\left[dt + 4n\sin^2\frac{\theta}{2}d\phi\right]^2 + \frac{1}{f(r)}dr^2 + (r^2 + n^2)(d\theta^2 + \sin^2\theta d\phi^2) \tag{39}$$

where,

$$f(r) = \frac{r^2 - 2Mr - n^2}{r^2 + n^2} \tag{40}$$

Here, M represents the "gravitoelectric mass" or 'mass' and n represents the "gravitomagnetic mass" or 'dual' (or 'magnetic') mass of this spacetime. It is obvious that the spacetime (39) is not invariant under time reversal $t \to -t$, signifying that it must have a sort of 'rotational sense', once again analogous to a magnetic monopole in electrodynamics. One is thus led to the conclusion that the source of the nonvanishing LT precession is this "rotational sense" arising from a nonvanishing NUT charge. Without the NUT charge, the spacetime is clearly hypersurface orthogonal and frame-dragging effects vanish, as already mentioned above. This *'dual' mass* has been investigated in detail in ref. [24, 25], who also refer to it as an *'angular momentum monopole* [16] in Taub-NUT spacetime. This implies that the inertial frame dragging seen here in such a spacetime can be identified as a *gravitomagnetic* effect.

In the Schwarzschild coordinate system, $f(r) = 0$ at

$$r = r_{\pm} = M \pm \sqrt{M^2 + n^2} \tag{41}$$

$r_{\pm}$ are similar to *horizons* in this geometry in the sense that $f(r)$ changes sign from positive to negative across the horizon and the radial coordinate r changes from spacelike to timelike. But is $r = r_+$ an event horizon in the sense of the event horizon of Schwarzschild spacetime ? We shall focus on this issue momentarily. For the present, we note that the LT precession rate (which can be easily obtained from eqn.(38) also) is given by [1]

$$\vec{\Omega}_{LT}^{MTN} = \frac{n(r^2 - 2Mr - n^2)^{\frac{1}{2}}}{(r^2 + n^2)^{\frac{3}{2}}}\hat{r} \tag{42}$$

---

[1] If we take the Taub-NUT metric(eqn.39) in more general form

$$ds^2 = -f(r)\left[dt - 2n(\cos\theta + C)d\phi\right]^2 + \frac{1}{f(r)}dr^2 + (r^2 + n^2)(d\theta^2 + \sin^2\theta d\phi^2)$$

our final result for LT precession rate(eqn.42) in Taub-NUT spacetime remains same. Because, this final result is independent of C which is a constant. In the eqn.39, we take the form of the metric[23] by Misner ($C = -1$).
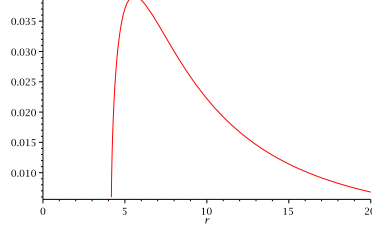
FIG. 1. Plot of Ω_{LT}^{MTN} vs r for $n = 3 \ m$ & $M = 1 \ m$

It is clear that $\Omega_{LT}^{MTN} = 0$ on $r = r_\pm$, in contrast to the LT precession frequency in the standard Kerr spacetime which is maximum closest to the event horizon ! Further, if we plot the magnitude of the precession rate as a function of the radial coordinate for $r \geq r_+$, as obtained from (42), one obtains the profile like FIG. 1.

Thus, the precession rate is maximum very close to the 'horizon' $r = r_+$, but it sharply drops for $r \to r_+$, most likely becoming ill-defined on the 'horizon'.

As the metric (39) blows up at $r = r_\pm$, we should perhaps try a different co-ordinate system where it is smooth on the 'horizon'. Following [26], wherein an analytic extension of the metric (39) has been attempted, one obtains the transformed metric

$$ds^2 = (r^2 + n^2)(d\theta^2 + \sin^2\theta d\phi^2)$$
$$+ F^2 \left[du_\pm^2 - dv_\pm^2 - (2n/r_\pm)(u_\pm dv_\pm - v_\pm du_\pm)\cos\theta d\phi - (n/r_\pm)^2(u_\pm^2 - v_\pm^2)\cos^2\theta d\phi^2 \right] \tag{43}$$

where,

$$F^2 = 4r_\pm^4(r^2 + n^2)^{-1}\left(\frac{r - r_\mp}{r_\pm}\right)^{1 - \frac{r_\mp}{r_\pm}}\exp\left(-\frac{r}{r_\pm}\right) \tag{44}$$

$$u_\pm = \left(\frac{r - r_\pm}{r_\pm}\right)^{1/2}\left(\frac{r - r_\mp}{r_\pm}\right)^{\frac{r_\mp}{2r_\pm}}\exp\left(\frac{r}{2r_\pm}\right)\cosh\left(\frac{t}{2r_\pm}\right) \tag{45}$$

$$v_\pm = \left(\frac{r - r_\pm}{r_\pm}\right)^{1/2}\left(\frac{r - r_\mp}{r_\pm}\right)^{\frac{r_\mp}{2r_\pm}}\exp\left(\frac{r}{2r_\pm}\right)\sinh\left(\frac{t}{2r_\pm}\right) \tag{46}$$

In u, v co-ordinate system r could be redefined as

$$u_\pm^2 - v_\pm^2 = \left(\frac{r - r_\pm}{r_\pm}\right)\left(\frac{r - r_\mp}{r_\pm}\right)^{\frac{r_\mp}{r_\pm}}\exp\left(\frac{r}{r_\pm}\right) \tag{47}$$

Recall that locally every spherically symmetric four dimensional spacetime has the structure $\mathcal{I}_2 \otimes S^2$ where $\mathcal{I}_2$ is a two dimensional Lorentzian spacetime. In this Taub-NUT case, the attempted analytic extension discussed immediately above leads to a *vanishing* of the two dimensional Lorentzian metric on the 'horizon' $r = r_+$, in contrast to the Schwarzschild metric. This might be taken to imply that perhaps the null surface $r = r_+$ is *not quite* an event horizon; rather it is a null surface where ingoing future-directed null geodesics appear to *terminate*, as already noticed in ref.s [6] and also earlier in [27]. So, physical effects on this null hypersurface might not be easy to compute, as a result of which the apparent vanishing of the LT precession on this hypersurface is to be taken with a pinch of salt.

The NUT spacetime, for the mass $M = 0$ is also well-defined (see, for example, appendix of [16]). We can also write down the precession rate only for massless dual mass (NUT charge n can be regarded as dual mass) solutions of NUT spacetime. This turns out to be

$$\vec{\Omega}_{LT}^{TN} = \frac{n(r^2 - n^2)^{\frac{1}{2}}}{(r^2 + n^2)^{\frac{3}{2}}}\hat{r} \tag{48}$$

At, the points $r = \pm n$, the LT precession vanishes akin to the previous case, but the same caveats apply here as well. One may plot the precession frequency as a function of the radial coordinate as earlier : Here, we observe that for $n = 3$, the LT precession is starting for $r > 3$ and continued to the infinity. Setting $\frac{d\Omega_{LT}^{TN}}{dr} = 0$, we get that Ω_{LT}^{TN} is maximum at $r = \sqrt{2}n$. In our figure (FIG. 2.), this value is $r = 3\sqrt{2} = 4.24$ m. Now, we are not interested for $r < 3$. Our formulas are not comfortable in that regions and $r < r_\pm$ is also not well-defined for Taub-NUT spacetimes. From our precession rate formulas (38,42,48) at dual mass spacetimes we can see
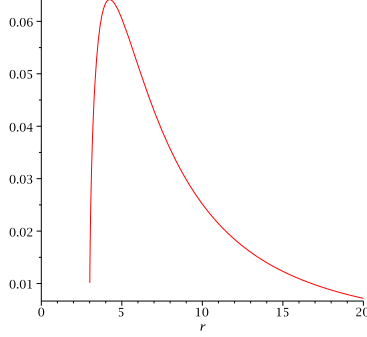
FIG. 2. Plot of Ω_{LT}^{TN} vs r for $n = 3$ m & $M = 0$

that the precession rate (Ω_{LT}^{TN}) is the same, starting from the polar region to the equatorial plane for a fixed distance. Ω_{LT}^{TN} depends only on distance (r) of the test gyroscope from the 'dual mass'.

In summary, we have noted in this subsection several subtleties of computing the LT precession rate on and near the 'horizon' of a Taub-NUT spacetime, and our results are consistent with earlier literature where geodesic incompleteness on this null hypersurface has been noted.

VII.   SUMMARY AND DISCUSSION

The analyses presented above has two important features : (a) the LT precision frequency of a gyroscope in a 'Copernican' frame within a Kerr spacetime is computed without any assumption on the angular momentum parameter or indeed the curvature of spacetime. The only comparable attempt in the literature is that in ref. [11], which however is not the same computation as ours, and the result is not the same in terms of metric coefficients. (b) The result derived in eqn. (4) is in fact valid, not just for axisymmetric spacetimes, but also for general non-static stationary spacetimes, once again without any assumptions about the curvatures involved. This result, we believe is applicable to a very large class of strong gravity systems.

The substantial difference between the LT precession frequency, arising in strong gravity regime and the standard, weak field precession rate for inertial frame dragging ought to provide a strong motivation for their measurement in space probes planned for the near future. The fascinating world of gravitational effects associated with strongly gravitating compact objects may provide the best yet dynamical observational signatures of general relativity.

At this point we refer to a recent paper by V. Kagramanova et al. [27] where it is claimed that the Lense-Thirring effect in fact vanishes *everywhere/everywhen* in the Taub-NUT spacetime. In that paper, timelike geodesic equations in this spacetime are investigated. The *orbital plane precession frequency* $(\Omega_\phi - \Omega_\theta)$ is computed, following the earlier work of ref. [28, 29], and a vanishing result ensues. This result is then interpreted in ref. [27] as a signature for a null LT precession in the Taub-NUT spacetime.

We would like to submit that what we have focused on in this paper is quite different from the 'orbital plane precession' considered in [27]. Using a 'Copernican' frame, we calculate the precession of a gyroscope which is moving in an arbitrary integral curve (not necessarily geodesic). Within this frame, an untorqued gyro in a stationary but not static spacetime held fixed by a support force applied to its center of mass, undergoes LT precession. Since the Copernican frame does not rotate (by construction) relative to the inertial frames at asymptotic infinity ("fixed stars"), the observed precession rate in the Copernican frame also gives the precession rate of the gyro relative to the fixed stars. It is thus, more an intrinsic property of the classical *spin* of the spacetime (as an untorqued gyro must necessarily possess), in the sense of a dual mass, rather than an *orbital* plane precession effect for timelike geodesics in a Taub-NUT spacetime. The dual mass is like the *Saha spin* of a magnetic monopole in electrodynamics [21], which may have a vanishing orbital angular momentum, but to which a spinning electron must respond in that its wavefunction acquires a geometric phase.

More specifically, in our case, we consider the gyroscope equation [14] in an arbitrary integral curve

$$\nabla_u S = <S, a> u \tag{49}$$

where, $a = \nabla_u u$ is the acceleration, u is the four velocity and S indicates the spacelike classical spin four vector $S^\alpha = (0, \vec{S})$ of the gyroscope. For geodesics $a = 0 \Rightarrow \nabla_u S = 0$.

In contrast, Kagramanova et al. [27] consider the behaviour of massive test particles with *vanishing spin* $S = 0$ [30], and compute the orbital plane precession rate for such particles, obtaining a vanishing result. We

are thus led to conclude that because two different situations are being considered, there is no inconsistency between our results and theirs.

We may reiterate that the preliminary numerical results on LT precession rates for various systems is intended to serve as motivation for prospective measurements in strong gravity situations, as also for further theoretical work towards understanding the emission mechanism of pulsars and x-ray emission from black holes and neutron stars. We expect nontrivial modifications to arise from incorporation of frame-dragging effects in the theoretical analyses of these phenomena. We hope to report on this in the near future.

There are additional avenues of further work currently being explored : the most general axisymmetric solution of Einstein's equation given by the Plebański-Demiański metric is being investigated for an understanding of the LT precession in this case [31].

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